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Existence and Nonexistence of Positive Solutions of a Nonlinear Third Order Boundary Value Problem

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This paper is dedicated to Miklos Farkas on the occasion of his seventy-fifth birthday.

Abstract

The authors consider a two-point third order boundary value problem, the motivation for which arises from the study of the beam equation. Sufficient conditions for the existence and nonexistence of positive solutions for the problems are obtained. An example is included to illustrate the results.

This paper is in final form and no version of it will be submitted for publication elsewhere.

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1 Introduction

We wish to consider the third order nonlinear two point boundary value problem

$$\begin{cases} u'''(t) = g(t)f(u(t)), \quad 0 < t < 1, \\ \\ u(0) = u(1) = u''(1) = 0. \end{cases} \tag{P}$$

We will assume throughout that

(H) $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions with $g(t) \not\equiv 0$ on $[0, 1]$.

We are interested in obtaining sufficient conditions for the existence and nonexistence of positive solutions to the problem (P). By a *positive solution* of (P), we mean a solution $u(t)$ such that $u(t) > 0$ for $t \in (0, 1)$.

The motivation for this problem is that of the deformation of an elastic beam that is clamped at one end ($t = 0$), and is supported by a sliding clamp at the other end ($t = 1$). This situation is described by the boundary value problem

$$\begin{cases} u''''(t) = g(t)f(u(t)), \quad 0 < t < 1, \\ \\ u(0) = u'(0) = u'(1) = u'''(1) = 0. \end{cases} \tag{B}$$

Problems for the beam equation involving sliding clamp boundary conditions have been considered, for example, by Collatz [6, §5.7] and more recently by Gupta [13]. If we let $v(t) = u'(t)$, then problem (B) above can be written as

$$v'''(t) = g(t)f\left(\int_0^t v(s)ds\right), \quad 0 < t < 1,$$

$$v(0) = v(1) = v''(1) = 0,$$

which we see has the form of our problem (P).

For a discussion of applications of boundary value problems to a variety of physical problems, we suggest the works of Bisplinghoff and Ashley [5], Fung [8], Love [17], Prescott [19], and Timoshenko [22] on elasticity, the monographs by Mansfield [18] and Soedel [20] on deformation of structures, and the work of Dulácska [7] on the effects of soil settlement. Excellent surveys of theoretical results can be found in Agarwal [1] and Agarwal, O'Regan, and Wong [2]. Some recent contributions to the study of third order nonlinear boundary value problems include, for example, the papers of Anderson

[3], Anderson and Davis [4], Kong et al. [14], Li [16], Sun [21], Yao [23], and the present authors [9, 10, 11].

To prove our results, we will use the following theorem, which is known as the Guo-Krasnosel'skii fixed point theorem [12, 15].

Theorem 1.1 *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{X}$ be a cone in $\mathcal{X}$. Assume that Ω_1, Ω_2 are bounded open subsets of $\mathcal{X}$ with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$\mathcal{L} : \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(K1) $\|\mathcal{L}u\| \leq \|u\|$ *if* $u \in \mathcal{P} \cap \partial\Omega_1$*, and* $\|\mathcal{L}u\| \geq \|u\|$ *if* $u \in \mathcal{P} \cap \partial\Omega_2$*; or*

(K2) $\|\mathcal{L}u\| \geq \|u\|$ *if* $u \in \mathcal{P} \cap \partial\Omega_1$*, and* $\|\mathcal{L}u\| \leq \|u\|$ *if* $u \in \mathcal{P} \cap \partial\Omega_2$*.*

Then $\mathcal{L}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

We choose $\mathcal{X} = C[0,1]$ with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \quad v \in \mathcal{X},$$

to be our Banach space. We also define the constants

$$F_0 = \limsup_{x\to 0^+} (f(x)/x), \quad f_0 = \liminf_{x\to 0^+} (f(x)/x),$$

$$F_\infty = \limsup_{x\to +\infty} (f(x)/x), \quad f_\infty = \liminf_{x\to +\infty} (f(x)/x).$$

The next section contains our existence results; Section 3 contains our nonexistence results as well as an example.

2 Green Function

The Green function for the problem consisting of the equation

$$u'''(t) = 0$$

and the boundary conditions in (P), namely,

$$u(0) = u(1) = u''(1) = 0, \tag{1}$$

is given by

$$G(t,s)=\begin{cases} t(2s-t-s^2)/2, & t\le s,\\ \\ s^2(1-t)/2, & s\le t.\end{cases}$$

It is easy to see that $G(t,s)\ge 0$ if $t,s\in[0,1]$ and strict inequality holds in the open interval. The problem (P) is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)g(s)f(u(s))ds,\quad 0\le t\le 1, \tag{I}$$

in the sense that if u is a solution of the boundary value problem (P), then it is a solution of the integral equation (I), and conversely.

Our first lemma provides information about functions that satisfy condition (1).

Lemma 2.1 *If $u\in C^3[0,1]$ satisfies* (1) *and*

$$u'''(t)\ge 0 \quad \textit{for} \quad 0\le t\le 1, \tag{2}$$

then $u(t)\ge 0$ on $[0,1]$.

Proof. The lemma follows easily from the fact that $G(t,s)\ge 0$ for t, $s\in[0,1]$. □

In the remainder of the paper, we let

$$a(t)=\min\{(1-t),3t\}.$$

Theorem 2.2 *The Green function $G(t,s)$ has the following properties.*

$$G(t,s)\ge 4tG(1/4,s),\quad \textit{for } 0\le t\le 1/4,\ 0\le s\le 1. \tag{3}$$

$$G(t,s)\ge (4/3)(1-t)G(1/4,s),\quad \textit{for } 1/4\le t\le 1,\ 0\le s\le 1. \tag{4}$$

$$G(t,s)\le (4/3)G(1/4,s),\quad \textit{for } 0\le t\le 1,\ 0\le s\le 1. \tag{5}$$

$$G(t,s)\ge (4/3)a(t)G(1/4,s),\quad \textit{for } 0\le t\le 1,\ 0\le s\le 1. \tag{6}$$

Proof. A little algebra is needed to prove this lemma. To prove (3), first note that

$$G(1/4,s)=\begin{cases} (2s-s^2-1/4)/8, & 1/4\le s,\\ \\ 3s^2/8, & s\le 1/4.\end{cases}$$

If $0\le s\le t\le 1/4$, then

$$G(t,s)-4tG(1/4,s)=s^2(1-4t)/2\ge 0.$$

If $0 \le t \le s \le 1/4$, then

$$G(t,s) - 4tG(1/4,s) = t\,[(s-t) + s(1-4s)]/2 \ge 0.$$

If $0 \le t \le 1/4 \le s \le 1$, then

$$G(t,s) - 4tG(1/4,s) = t\,(1/4-t)/2 \ge 0.$$

Thus, (3) is proved.

Next, we prove (4). If $s \le 1/4 \le t$, then

$$G(t,s) - (4/3)(1-t)G(1/4,s) = 0.$$

If $1/4 \le s \le t$, then

$$G(t,s) - (4/3)(1-t)G(1/4,s) = (1/6)(1-t)(2s-1/2)^2 \ge 0.$$

If $1/4 \le t \le s$, then

$$\begin{aligned} G(t,s) - (4/3)(1-t)G(1/4,s) &= (1/24)(4t-1)(8s-4s^2-3t-1) \\ &= (1/24)(4t-1)[(4s-1)(1-s)+3(s-t)] \ge 0. \end{aligned}$$

Thus, (4) holds.

In order to prove (5), we first see that if $t \le s \le 1/4$, then

$$(4/3)G(1/4,s) - G(t,s) = \frac{1}{2}(s-t)^2 + \frac{1}{2}ts^2 \ge 0.$$

If $t \le s$ and $1/4 \le s$, then

$$\begin{aligned} (4/3)G(1/4,s) - G(t,s) &= \frac{1}{6}(2s-1/4-s^2-6ts+3t^2+3ts^2) \\ &= \frac{1}{6}[3(t-s+s^2/2)^2 + (1/4)(6s-3s^2-1)(s-1)^2] \ge 0. \end{aligned}$$

In the last inequality, we used the fact that $6s-3s^2-1 \ge 0$ for $1/4 \le s \le 1$. If $s \le t$ and $s \le 1/4$, then

$$(4/3)G(1/4,s) - G(t,s) = \frac{1}{2}ts^2 \ge 0.$$

If $1/4 < s \le t$, then

$$\begin{aligned} (4/3)G(1/4,s) - G(t,s) &= \frac{1}{6}(2s - \frac{1}{4} - 4s^2 + 3ts^2) \\ &= \frac{1}{6}(t-1/4)^{-1}\,[(3/16)t(t-s) \\ &\quad +(2t-1/4-4t^2+3t^3)(s-1/4) \\ &\quad +(1/16)(4t-1)(4-3t)(4s-1)(t-s)] \\ &\ge 0, \end{aligned}$$

where in the last inequality, we used the fact that $2t-1/4-4t^2+3t^3 \geq 0$ for $1/4 \leq t \leq 1$. This proves (5).

Finally, it is clear that (6) follows immediately from (3) and (4). $\square$

Next, we obtain an important estimate on functions satisfying (1) and (2).

Theorem 2.3 *If $u \in C^3[0,1]$ satisfies (1) and (2), then $u(t) \geq a(t)\|u\|$ on $[0,1]$.*

Proof. Suppose that $u(t)$ achieves its maximum at $t = t_0$, that is, $\|u\| = u(t_0)$. Then, from (6) and (5), we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)u'''(s)ds \\ &\geq (4/3)\int_0^1 a(t)G(1/4,s)u'''(s)ds \\ &\geq (4/3)a(t)\int_0^1 (3/4)G(t_0,s)u'''(s)ds \\ &= a(t)u(t_0) \\ &= a(t)\|u\|. \end{aligned}$$

This completes the proof. $\square$

The next theorem is an immediate consequence of Theorem 2.3.

Theorem 2.4 *If $u \in C^3[0,1]$ is a nonnegative solution of the problem (P), then $u(t) \geq a(t)\|u\|$ on $[0,1]$.*

3 Existence of Positive Solutions

We need to define the constants A and B by

$$A = \int_0^1 G(1/4,s)g(s)a(s)\,ds \quad \text{and} \quad B = (4/3)\int_0^1 G(1/4,s)g(s)\,ds,$$

and we let

$$\mathcal{P} = \{v \in \mathcal{X} \mid v(t) \geq a(t)\|v\| \ \text{ on } \ [0,1]\}.$$

Clearly, $\mathcal{P}$ is a positive cone of the Banach space $\mathcal{X}$. Define an operator $\mathcal{T}: \mathcal{P} \to \mathcal{X}$ by

$$\mathcal{T}u(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \leq t \leq 1.$$

It is well known that $\mathcal{T} : \mathcal{P} \to \mathcal{X}$ is a completely continuous operator. Moreover, by the same type of argument as the one used in the proof of Theorem 2.3, we can show that $\mathcal{T}(\mathcal{P}) \subset \mathcal{P}$.

Now the integral equation (I) is equivalent to the equality

$$u = \mathcal{T}u, \quad u \in \mathcal{P}.$$

Thus, in order to obtain a positive solution of the problem (P), we only need to find a fixed point of $\mathcal{T}$ in $\mathcal{P}$. Our first existence result is the following.

Theorem 3.1 *If $BF_0 < 1 < Af_\infty$, then the boundary value problem (P) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \leq 1$. There exists $H_1 > 0$ such that

$$f(x) \leq (F_0 + \varepsilon)x \ \text{ for } \ 0 < x \leq H_1.$$

If $u \in \mathcal{P}$ and $\|u\| = H_1$, then for $0 \leq t \leq 1$, we have

$$\begin{aligned}
(\mathcal{T}u)(t) &= \int_0^1 G(t,s)g(s)f(u(s))\,ds \\
&\leq (4/3)\int_0^1 G(1/4,s)g(s)(F_0+\varepsilon)u(s)\,ds \\
&\leq (4/3)(F_0+\varepsilon)\|u\| \int_0^1 G(1/4,s)g(s)ds \\
&= (F_0+\varepsilon)\|u\|B \\
&\leq \|u\|,
\end{aligned}$$

which means $\|\mathcal{T}u\| \leq \|u\|$. So, if we let

$$\Omega_1 = \{u \in \mathcal{X} \mid \|u\| < H_1\},$$

then

$$\|\mathcal{T}u\| \leq \|u\| \ \text{ for } \ u \in \mathcal{P} \cap \partial\Omega_1.$$

Now choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta)\int_c^{1-c} G(1/4,s)g(s)a(s)\,ds > 1.$$

There exists $H_3 > 0$ such that

$$f(x) \geq (f_\infty - \delta)x \text{ for } x \geq H_3.$$

Let $H_2 = H_3/c + H_1$. If $u \in \mathcal{P}$ with $\|u\| = H_2$, then by Theorem 2.4, for $c \le t \le 1-c$, we have

$$u(t) \ge \min\{t, 1-t\}\|u\| \ge cH_2 \ge H_3.$$

So, if $u \in \mathcal{P}$ with $\|u\| = H_2$, then

$$\begin{aligned}
(\mathcal{T}u)(1/4) &\ge \int_c^{1-c} G(1/4,s)g(s)f(u(s))ds \\
&\ge \int_c^{1-c} G(1/4,s)g(s)\,(f_\infty - \delta)u(s)\,ds \\
&\ge (f_\infty - \delta)\|u\| \int_c^{1-c} G(1/4,s)g(s)a(s)\,ds \\
&\ge \|u\|,
\end{aligned}$$

which means $\|\mathcal{T}u\| \ge \|u\|$. So, if we let

$$\Omega_2 = \{u \in \mathcal{X} \mid \|u\| < H_2\},$$

then

$$\|\mathcal{T}u\| \ge \|u\| \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_2.$$

Thus, condition (K1) of Theorem 1.1 is satisfied, and so there exists a fixed point of $\mathcal{T}$ in $\mathcal{P}$. This completes the proof of the theorem. □

Remark 3.2 The condition $BF_0 < 1 < Af_\infty$ in Theorem 3.1 (also see Theorem 3.3 below) has a form similar to those found in many other papers on existence of positive solutions of nonlinear boundary value problems. Inherent in all such problems is of course the boundary conditions used since that determines the form of the Green function $G(t,s)$ and in turn the values of the constants A and B above. What determines the sharpness of the results, however, is the ability to estimate the positive solutions by constructing appropriate functions like $a(t)$ (see Theorems 2.2 and 2.4) used in defining the constants A and B and in the definition of the cone $\mathcal{P}$.

We also have the following companion result.

Theorem 3.3 *If $BF_\infty < 1 < Af_0$, then the boundary value problem (P) has at least one positive solution.*

Proof. Choose $\varepsilon > 0$ such that $(f_0 - \varepsilon)A \ge 1$. There exists $H_1 > 0$ such that

$$f(x) \ge (f_0 - \varepsilon)x \quad \text{for} \quad 0 < x \le H_1.$$

So, for $u \in \mathcal{P}$ with $\|u\| = H_1$, we have

$$\begin{aligned}
(\mathcal{T}u)(1/4) &= \int_0^1 G(1/4,s)g(s)f(u(s))\,ds \\
&\geq (f_0 - \varepsilon)\int_0^1 G(1/4,s)g(s)u(s)\,ds \\
&\geq (f_0 - \varepsilon)\|u\|\int_0^1 G(1/4,s)g(s)a(s)\,ds \\
&= (f_0 - \varepsilon)\|u\|A \\
&\geq \|u\|,
\end{aligned}$$

which means $\|\mathcal{T}u\| \geq \|u\|$. Hence, if we let

$$\Omega_1 = \{u \in \mathcal{X} \mid \|u\| < H_1\},$$

then

$$\|\mathcal{T}u\| \geq \|u\| \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_1.$$

Next, we choose $\delta \in (0,1)$ such that $((F_\infty + \delta)B + \delta) < 1$. There exists an $H_3 > 0$ such that

$$f(x) \leq (F_\infty + \delta)x \quad \text{for} \quad x \geq H_3.$$

Let $M = \max\limits_{0 \leq x \leq H_3} f(x)$ and

$$K = \frac{4M}{3}\int_0^1 G(1/4,s)g(s)\,ds.$$

Then,

$$f(x) \leq M + (F_\infty + \delta)x \quad \text{for} \quad x \geq 0.$$

Let $H_2 = H_1 + K/(1-(F_0+\delta)B)$. Then $H_2 > H_1$. If $u \in \mathcal{P}$ such that $\|u\| = H_2$, then we have

$$\begin{aligned}
(\mathcal{T}u)(t) &\leq \int_0^1 G(t,s)g(s)f(u(s))\,ds \\
&\leq (4/3)\int_0^1 G(1/4,s)g(s)f(u(s))\,ds \\
&\leq (4/3)\int_0^1 G(1/4,s)g(s)(M + (F_\infty + \delta)u(s))\,ds \\
&\leq K + (4/3)\int_0^1 G(1/4,s)g(s)(F_\infty + \delta)u(s)\,ds \\
&\leq K + (4/3)(F_\infty + \delta)\|u\|\int_0^1 G(1/4,s)g(s)\,ds \\
&= K + (F_\infty + \delta)\|u\|B \\
&\leq H_2,
\end{aligned}$$

which means $\|\mathcal{T}u\| \leq \|u\|$. So, if we let

$$\Omega_2 = \{u \in \mathcal{X} \mid \|u\| < H_3\},$$

then

$$\|\mathcal{T}u\| \leq \|u\| \quad \text{for} \quad u \in \mathcal{P} \cap \partial\Omega_2.$$

Now from Theorem 1.1, we see that the problem (P) has at least one positive solution, and this completes the proof of the theorem. □

4 Nonexistence Results and an Example

In this section, we present our nonexistence results as well as an example of our theorems. Note that condition (H) holds throughout this section as well.

Theorem 4.1 *If $Bf(x) < x$ for all $x \in (0, +\infty)$, then the problem (P) has no positive solutions.*

Proof. Assume, to the contrary, that $x(t)$ is a positive solution of (P). Then

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)g(s)f(x(s))\,ds \\ &< B^{-1}\int_0^1 G(t,s)g(s)x(s)\,ds \\ &\leq \frac{4}{3B}\|x\|\int_0^1 G(1/4,s)g(s)ds \\ &= \|x\|, \end{aligned}$$

which is a contradiction. □

The proof of the following theorem is similar to the one above and we omit the details.

Theorem 4.2 *If $Af(x) > x$ for all $x \in (0, +\infty)$, then the problem (P) has no positive solutions.*

We illustrate our results with the following example.

Example 4.3 Consider the third order boundary value problem

$$\begin{cases} u'''(t) = \lambda(6-5t)\frac{u(1+6u)}{(1+u)}, & 0 < t < 1, \\ u(0) = u(1) = u''(1) = 0. \end{cases} \tag{E}$$

We see that $F_0 = f_0 = \lambda$ and $F_\infty = f_\infty = 6\lambda$. Calculations indicate that

$$A = 477/8192, \quad B = 99/512.$$

From Theorem 3.1, we see that if

$$2.8624 \approx 1/(6A) < \lambda < 1/B \approx 5.1717,$$

then problem (E) has at least one positive solution. From Theorems 4.1 and 4.2, we see that if

$$\lambda < 1/(6B) \approx 0.8619 \quad \text{or} \quad \lambda > 1/A \approx 17.1741,$$

then the problem (E) has no positive solutions.

In conclusion, we would like to point out that we have not required that $f_0 = F_0 = 0$ and $f_\infty = F_\infty = +\infty$ (f is superlinear), or that $f_0 = F_0 = +\infty$ and $f_\infty = F_\infty = 0$ (f is sublinear), or even that $f(u)/u$ has limit at 0 or ∞. However, if f is superlinear, then Theorem 3.1 applies, while if f is sublinear, then Theorem 3.3 should be used. Also, we do not ask that $g(t)$ not vanish identically on any subinterval of $[0, 1]$ as is often done.

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